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Application of Elzaki Transform Decomposition Method in Solving Time-Fractional Sawada Kotera Ito Equation

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Abstract

The research paper primarily focuses on the theoretical framework and mathematical methodology for solving the Sawada Kotera Ito with time fraction (SKI) equation using the Elzaki transform decomposition technique, including the Caputo-Fabrizio derivative and the Atangana-Baleanu derivative, which are crucial for comprehending the fractional SKI (1). Various numerical examples are given to illustrate the application of the Elzaki transform in solving the fractional SKI equation. We utilize efficient basis functions, namely fractional Lagrange functions, for the interpolation of temporal variables. By integrating the primary equation with the initial-boundary conditions and employing the relevant operational matrices for spatial and fractional temporal variables, the proposed model is converted into a system of nonlinear algebraic equations, which can be resolved using effective iterative solvers. Furthermore, we conduct a comprehensive analysis of the method's convergence. Furthermore, we evaluate multiple test issues to assess the proposed scheme, which demonstrates its superior accuracy and less computational expense compared to contemporary numerical methods in the literature.

Keywords: Elzaki transform decomposition; Atangana-Baleanu fractional derivative; Ito equation of Sawada Kotera Caputo derivative and Caputo-Fabrizio derivative.

1 Introduction

Integral transformations are advantageous for their simplifying capabilities and are frequently employed in addressing differential equations with particular boundary conditions. The judicious use of integral transformations facilitates the conversion of differential and integral equations into a solvable algebraic equation structure.The attained solution is, naturally, the transformation of the original differential equation's solution, and it is essential to invert this transformation to finalize the procedure [18, 13].

Over the past two decades, numerous integral transformations have been developed within the realm of Laplace transforms, including the Smudu, Elzaki, Tabbi, Abboud, Borreza, Muhannad, $G_{transform}$, Sawy, and Kamal transforms. These transformations have been employed to resolve many forms of integral equations, such as Ordinary Differential Equations (ODEs), Partial Differential Equations (PDEs), and Fractional Differential Equations (FDEs) [12].The integration of these transformations with additional techniques, including Adomian analysis and homotopy perturbation approaches has been employed to address various types of ODEs, PDEs, and FDEs [20, 15].

In this study, we employed the Tweel-Elazki approach due to its efficacy in addressing the complexity inherent in fractional differential equations. Fractional differential equations necessitate specialized procedures and precise techniques to yield accurate and dependable outcomes [17]. This drives our effort to examine an efficient technique, the Elzaki Transform, which converts the original issue into a more tractable algebraic format, facilitating the derivation of analytical solutions. The Elzaki integral transform represents the most sophisticated integral transform inside the Laplace category, with all current integral transforms being specific instances of this transform [19].

The subsequent structure of the paper is outlined as follows Section 2 presents many fundamental definitions and theorems of our suggested technique. In Section 3, we implement the transformation on the SKI equation, detailing the step-by-step procedure for deriving the modified equation and obtaining the solutions [9]. Section 4 conducts a thorough convergence analysis to assess the validity and stability of the solutions obtained from this method, while Section 5 illustrates the efficacy of the proposed methodology through numerical applications, confirming its correctness and reliability. Section 6 offers a concise conclusion.

$$\begin{aligned} D_t^{\alpha}\mathrm{u}(\gamma,t) = &-252\mathrm{u}^3\mathrm{u}_\gamma - 63\mathrm{u}_\gamma^3 - 378\mathrm{u}\mathrm{u}_\gamma\mathrm{u}_{\gamma\gamma} - 126\mathrm{u}^2\mathrm{u}_{\gamma\gamma\gamma} \\ &- 63\mathrm{u}_{\gamma\gamma}\mathrm{u}_{\gamma\gamma\gamma} - 42\mathrm{u}_\gamma\mathrm{u}_{\gamma\gamma\gamma\gamma} - 21\mathrm{u}\mathrm{u}_{\gamma\gamma\gamma\gamma\gamma} - \mathrm{u}_{\gamma\gamma\gamma\gamma\gamma\gamma\gamma}, \end{aligned} \tag{1}$$

with the starting condition,

$$\mathrm{u}(\gamma,0) = \frac{4}{3}b^2\left(2 - 3\tanh^2(b\gamma)\right). \tag{2}$$

This study focuses on using the Elzaki transform decomposition approach to solve the seventh-order time-fractional Sawada Kotera Ito equation, which has kernel derivatives that are both singular and non-singular [2]. The Elzaki transformation and decomposition approach are combined to create ETDM. The proposed technique reduces the complexity of estimating the series terms by eliminating the need to compute fractional derivatives or fractional integrals in the recursive method [1], compared to the traditional Adomian procedure. Round-off errors are avoided using ETDM, which also eliminates the need for linearization, prescribed assumptions, perturbation, and discretization. Nonlinear differential equations involving multiple variables, such as linear and nonlinear partial differential equations, are also addressed nonlinear.

2 Definitions and Properties

In this section, we will learn about the ET and fractional derivative, as well as some basic introductions and definitions. The text discusses the use of power-law and Mittag-Leffler functions in the fractional derivatives framework.

Definition 2.1. *Riemann-Liouville fractional integral (RLI) operator of order $\alpha > 0$ for a function $y(\tau)$ is given by,*

$$D^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} y^n(\tau) d\tau = I^{n-\alpha} y^n(t), \qquad t > 0. \tag{3}$$

Definition 2.2. [12] *For $y \in H^1(0,t)$, $t > 0$, $T > 0$, $\alpha \in (0,1]$. Then the CF fractional operator is given by,*

$$ {}_0^{CF}D_t^{\alpha} y(t) = \frac{B(\alpha)}{1-\alpha)} \frac{d}{dt} \int_0^t y(\tau) e^{-\alpha \frac{t-\tau}{1-\alpha}} d\tau, \qquad 0 < \alpha < 1.$$

Definition 2.3. [23] *Caputo derivative of order $0 \leq n-1 < \alpha < n$ with the lower limit zero for a function $y(\tau)$ is given by,*

$$I^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau, \qquad t > 0. \tag{4}$$

Definition 2.4. [14] *Let $0 < \alpha < 1$, denote the α order the A-B-fractional derivative defined as follows,*

$$ {}^{ABF}D_t^{\alpha} f(t) = \frac{\beta(1)}{1-\alpha} \int_0^t f'(\varsigma) E_{\alpha} \left(\frac{-\alpha(t-\varsigma)}{1-\alpha} \right) d\varsigma, \qquad t \geq 0, \qquad 0 < \alpha < 1, \tag{5}$$

where $f \in H'(0,T)$, function of normalization is $\beta(1)$, and the Mittag-Leffler function (5).

Definition 2.5. [10] *The Elzaki transform of a given mapping $f(t)$ is stated as following,*

$$E(f(t)) = T(v) = v \int_0^{\infty} e^{-\frac{t}{v}} f(t) dt, \qquad t \geq 0, \qquad v \in [p_1, p_2]. \tag{6}$$

Definition 2.6. [26] *The Elzaki transform of the Caputo - Fractional derivative is given as following,* (6)

$$E\left[{}^{cD}D_t^{\alpha} f(t) \right] = v^{-\alpha} T(v) - \sum_{k=0}^{i-1} v^{2-\alpha+k} f^k(0), \qquad i-1 < \alpha \leq i. \tag{7}$$

Definition 2.7. [22] *The Elzaki transform of the Caputo - Fabrizo fractional derivative of the function $f(t)$ of order α, where $0 < \alpha \leq 1$ and $n \in \square \cup \{0\}$ is given,*

$$E\left[{}^{CF}D_t^{\alpha} f(t) \right] = \frac{1}{1-\alpha(1-v)} \left[T(v) - v^2 f(0) \right]. \tag{8}$$

Definition 2.8. [4] *The Elzaki transform of the A-B-Caputo fractional derivative operator is given as following,*

$$E\left[{}^{ABC}D_t^{\alpha} f(t) \right] = \frac{N(\alpha)}{\alpha v^{\alpha} + 1 - \alpha} \left[v^{-1} T(v) - v f(0) \right]. \tag{9}$$

3 Methodology

In this section, we apply a unique approximation analytical approach to the following equation, which is generated from the Elzaki transform decomposition process,

$$D^{\alpha}u(x,t)+Ru(x,t)+Nu(x,t)=f(x,t), \qquad x,t\geq 0, \qquad m-1<\alpha<m, \tag{10}$$

where $D^{\alpha}=\dfrac{\partial^{\alpha}}{\partial^{\alpha}}$. The order α the Caputo fractional derivative is defined for natural numbers m, with R representing a linear operator [21], N denoting a nonlinear function, and f representing the source function. The initial and boundary conditions corresponding to (10) are expressed in the following format,

$$u(x,0)=h(x), \qquad 0<\alpha\leq 1, \qquad u(x,t)\to\infty, \qquad t>0, \tag{11}$$

and

$$u(x,0)=h(x), \qquad \frac{\partial u(x,0)}{\partial t}=k(x), \qquad 1<\alpha\leq 2, \qquad u(x,t)\to\infty, \qquad t>0.$$

Case I: (ETDM-C)

By applying the Caputo sense fractional derivative to the Elzaki transform of (10), we may get,

$$E\left[D^{\alpha}u(x,t)\right]+E[Ru(x,t)]+E[Nu(x,t)]=E[f(x,t)], \qquad \alpha>0.$$

Using the property of Elzaki transform to get,

$$\frac{u(x,v)}{v^{\alpha}}-C+E[Ru(x,t)]+E[Nu(x,t)]=E[f(x,t)], \qquad \alpha>0, \tag{12}$$

where $C=\sum_{k=0}^{n-1}v^{2-\alpha+k}u^{(k)}(x,0)$ and

$$E[u(x,t)]=v^{\alpha}E[f(x,t)]+v^{\alpha}C-v^{\alpha}E[Ru(x,t)]-v^{\alpha}E[Nu(x,t)]. \tag{13}$$

The stander the Elzaki decomposition technique provides the answer, $u(x,t)$ by the series,

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t). \tag{14}$$

The nonlinear operator is decomposition as,

$$Nu(x,t)=\sum_{n=0}^{\infty}A_n. \tag{15}$$

By applying the first Adomian polynomials [23] to the nonlinear function $Nu(x,t)$, we obtain,

$$E\left[\sum_{n=0}^{\infty}u_n(x,t)\right]=v^{\alpha}E[f(x,t)]+v^{\alpha}C-v^{\alpha}E\left[R\sum_{n=0}^{\infty}u_n(x,t)\right]-v^{\alpha}E\left[\sum_{n=0}^{\infty}A_n\right]. \tag{16}$$

Matching both sides of (16) yield the following iterative algorithm,

$$\begin{aligned} E\left[u_0(x,t)\right] &= v^{\alpha}E[f(x,t)] + v^{\alpha}C, \\ E\left[u_1(x,t)\right] &= -v^{\alpha}E\left[Ru_0(x,t)\right] - v^{\alpha}E\left[A_0\right], \end{aligned} \tag{17}$$

$$E\left[u_2(x,t)\right] = -v^{\alpha}E\left[Ru_1(x,t)\right] - v^{\alpha}E\left[A_1\right]. \tag{18}$$

Typically, the recursive relation is defined as,

$$E\left[u_{n+1}(x,t)\right] = -v^{\alpha}E\left[Ru_n(x,t)\right] - v^{\alpha}E\left[A_n\right], \qquad n \geq 1. \tag{19}$$

Applying inverse Elzaki transform (17)−(19) to obtain,

$$u_0(x,t) = H(t), \tag{20}$$

$$u_{n+1}(x,t) = -E^{-1}\left[v^{\alpha}E\left[Ru_n(x,t)\right] + v^{\alpha}E\left[A_n\right]\right], \qquad n \geq 1. \tag{21}$$

Case II: (ETDM-CF)

Using the CF fractional derivative and the Elzaki transform of (10), we get,

$$E\left[D^{\alpha}u(x,t)\right] + E[Ru(x,t)] + E[Nu(x,t)] = E[f(x,t)], \qquad \alpha > 0.$$

By utilizing the Elzaki transform property, we may obtain,

$$\frac{1}{1-\alpha(1-v)}\left[T(v) - v^2 f(0)\right] + E[Ru(x,t)] + E[Nu(x,t)] = E[f(x,t)], \qquad \alpha > 0,$$

where $C = \sum_{k=0}^{n-1} v^{2-\alpha+k}u^{(k)}(x,0)$ and

$$\begin{aligned} E[u(x,t)] = v^2C &+ \big(1-\alpha(1-v)\big)\big(E[f(x,t)]\big) \\ &- \big(1-\alpha(1-v)\big)\big(E[Ru(x,t)] + E[Nu(x,t)]\big). \end{aligned} \tag{22}$$

The stander Elzaki decomposition method Specifies the resolution The function $u(x,t)$ can be represented as a series,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{23}$$

The Nonlinear operator is decomposition as,

$$Eu(x,t) = \sum_{n=0}^{\infty} A_n. \tag{24}$$

By applying the first Adomian polynomials [23] to the nonlinear function $Nu(x,t)$, we obtain,

$$\begin{aligned} E\left[\sum_{n=0}^{\infty} u_n(x,t)\right] &= v^2C + (1-\alpha(1-v))(E[f(x,t)]) \\ &\quad - ((1-\alpha(1-v)))E\left(\left[R\sum_{n=0}^{\infty} u_n(x,t)\right] - \left[\sum_{n=0}^{\infty} A_n\right]\right). \end{aligned} \tag{25}$$

The iterative procedure obtained by equating both sides of (25) is as follows, as mentioned in [7],

$$E\left[u_0(x,t)\right] = v^2C + (1-\alpha(1-v))(E[f(x,t)]), \tag{26}$$

$$\begin{aligned} E\left[u_1(x,t)\right] &= -((1-\alpha(1-v)))E\left(Ru_0(x,t)+A_0\right), \\ E\left[u_2(x,t)\right] &= -((1-\alpha(1-v)))E\left(Ru_1(x,t)+A_1\right). \end{aligned} \tag{27}$$

In general, the recursive relation is given by,

$$E\left[u_{n+1}(x,t)\right] = -((1-\alpha(1-v)))E\left(Ru_n(x,t)+A_n\right), \qquad n \geq 1. \tag{28}$$

Applying inverse Elzaki transform (26)−(28), to obtain,

$$\begin{aligned} u_0(x,t) &= H(t), \\ u_{n+1}(x,t) &= -E^{-1}\Big[((1-\alpha(1-v)))E\left(Ru_n(x,t)+A_n\right)\Big], \qquad n \geq 1. \end{aligned} \tag{29}$$

Case III: (ETDM-ABC)

[6] Using the Atangana-Balena-Caputo fractional derivative and the Elzaki transform of (10), we get,

$$E\left[D^\alpha u(x,t)\right] + E[Ru(x,t)] + E[Nu(x,t)] = E[f(x,t)], \qquad \alpha > 0.$$

Using the property of Elzaki transform, to get,

$$\left(\frac{N(\alpha)}{\alpha v^\alpha + 1 - \alpha}\right)\left[v^{-1}T(v) - vf(0)\right] + E[Ru(x,t)] + E[Nu(x,t)] = E[f(x,t)], \qquad \alpha > 0,$$

where $C = \sum_{k=0}^{n-1} v^{2-\alpha+k}u^{(k)}(x,0)$ and [3],

$$\begin{aligned} E[u(x,t)] = v^2C &+ \left(\frac{\alpha v^\alpha + 1 - \alpha}{N(\alpha)}\right)(E[f(x,t)]) \\ &- \left(\frac{\alpha v^\alpha + 1 - \alpha}{N(\alpha)}\right)(E[Ru(x,t)] + E[Nu(x,t)]). \end{aligned}$$

The stander Elzaki decomposition method specifies the resolution. $u(x,t)$ by the series,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{30}$$

The nonlinear operator is decomposition as,

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n. \tag{31}$$

For the nonlinear function $Nu(x,t)$ the first Adomian polynomials [3] to get,

$$\begin{aligned} E\left[\sum_{n=0}^{\infty} u_n(x,t)\right] = v^2C &+ \left(\frac{\alpha v^\alpha + 1 - \alpha}{N(\alpha)}\right)(E[f(x,t)]) \\ &- \left(\frac{\alpha v^\alpha + 1 - \alpha}{N(\alpha)}\right)E\left(\left[R\sum_{n=0}^{\infty} u_n(x,t)\right] - \left[\sum_{n=0}^{\infty} A_n\right]\right). \end{aligned} \tag{32}$$

Matching both sides of (32) yield the following iterative algorithm [3],

$$E\left[u_0(x,t)\right] = v^2 C + \left(\frac{\alpha v^\alpha + 1 - \alpha}{N(\alpha)}\right)(E[f(x,t)]), \tag{33}$$

$$\begin{aligned} E\left[u_1(x,t)\right] &= -\left(\frac{\alpha v^\alpha + 1 - \alpha}{N(\alpha)}\right) E\left(Ru_0(x,t) + A_0\right), \\ E\left[u_2(x,t)\right] &= -\left(\frac{\alpha v^\alpha + 1 - \alpha}{N(\alpha)}\right) E\left(Ru_1(x,t) + A_1\right). \end{aligned} \tag{34}$$

In general, the recursive relation is given by,

$$E\left[u_{n+1}(x,t)\right] = -\left(\frac{\alpha v^\alpha + 1 - \alpha}{N(\alpha)}\right) E\left(Ru_n(x,t) + A_n\right), \qquad n \geq 1. \tag{35}$$

Applying inverse Elzaki transform (33)−(35), to obtain [16],

$$\begin{aligned} u_0(x,t) &= H(t), \\ u_{n+1}(x,t) &= -E^{-1}\left[\left(\frac{\alpha v^\alpha + 1 - \alpha}{N(\alpha)}\right) E\left(Ru_n(x,t) + A_n\right)\right], \qquad n \geq 1. \end{aligned} \tag{36}$$

4 Analysis of Convergence

In this part, we have discussed the convergence and uniqueness of the ETDM-ABC, ETDM-CF, and ETDM-C.

Theorem 4.1. *The* (10) ***ETDM-C*** *solution is distinct when,*

$$0 < (\Delta_1 + \Delta_2)\,\frac{u^\alpha}{\Gamma(1+\alpha)} < 1.$$

Proof. Let $F = (A[I], \|.\|)$ serve as the Banach space when the norm is $\vartheta(u)\| = \max t \in J|\vartheta(u)|, \forall$ functions that are continuous on J. Consider $G : F \to$. A nonlinear mapping is F, where $v_l^A(\gamma, u) = v_0^A$,

$$\begin{aligned} \mathfrak{v}^B(\gamma,\mathrm{t}) &= \mathfrak{v}_0^B(\gamma,\mathrm{v}) + \mathfrak{v}_1^B(\gamma,\mathrm{v}) + \mathfrak{v}_2^B(\gamma,\mathrm{v}) + \ldots, \\ \mathfrak{u}_{n+1}^A(\gamma,\mathrm{t}) &= \mathfrak{u}_0^A + \mathbf{ET}^{-1}\left[\left(\frac{v}{S}\right)^\alpha \mathbf{ET}\left[\mathcal{L}\left(\mathfrak{v}_l(\gamma,\mathrm{u})\right)\right]\right] + \mathbf{ET}^{-1}\left[\left(\frac{v}{S}\right)^\alpha \mathbf{ET}\left[\mathcal{E}\left(\mathfrak{v}_l(\gamma,\mathrm{u})\right)\right]\right], \qquad l \geq 0. \end{aligned} \tag{37}$$

Assume that,

$$|\backslash(\mathfrak{u}) - \backslash\,(\mathfrak{v}^*)| < \Delta_1\,|\mathfrak{v} - \mathfrak{u}^*| \qquad \text{and} \qquad |\mathcal{E}(\mathfrak{v}) - \mathcal{E}\,(\mathfrak{v}^*)| < \Delta_2\,|\mathfrak{v} - \mathfrak{v}^*|\,,$$

where Δ_1 and Δ_2 are Lipschitz constants and $\mathfrak{v} := \mathfrak{u}(\zeta, \mathrm{u})$ and $\mathfrak{v}^* := \mathfrak{t}^*(\gamma, \mathrm{t})$ These are two distinct values of a function.

$$\begin{aligned}
\|H\mathfrak{v}-H\mathfrak{u}^*\| &= \max_{\mathfrak{u}\in I}\left|\mathbf{ET}^{-1}\left[\left(\frac{l}{r}\right)^{\alpha}\mathbf{ET}[\mathcal{L}(\mathfrak{u})+\mathcal{E}(\mathfrak{t})]\right]-\mathbf{ET}^{-1}\left[\left(\frac{v}{r}\right)^{\alpha}\mathbf{ET}\left[\backslash\left(\mathfrak{v}^*\right)+\mathcal{E}\left(\mathfrak{v}^*\right)\right]\right]\right|\\
&\leq \max_{\mathfrak{t}\in J}\left|\mathbf{ET}^{-1}\left[\left(\frac{l}{r}\right)^{\alpha}\mathbf{ET}\left[\mathcal{L}(\mathfrak{v})-\backslash\left(\mathfrak{v}^*\right)\right]+\left(\frac{v}{r}\right)^{\alpha}\mathbf{ET}\left[\mathcal{E}(\mathfrak{v})-\mathcal{E}\left(\mathfrak{v}^*\right)\right]\right]\right|\\
&\leq \max_{\mathfrak{t}\in I}\left[\Delta_1\mathbf{ET}^{-1}\left[\left(\frac{l}{r}\right)^{\alpha}\mathbf{ET}\left|\mathfrak{v}-\mathfrak{v}^*\right|\right]+\Delta_2\mathbf{ET}^{-1}\left[\left(\frac{l}{r}\right)^{\alpha}\mathbf{ET}\left|\mathfrak{u}-\mathfrak{v}^*\right|\right]\right]\\
&\leq \max_{\mathfrak{u}\in J}\left(\Delta_1+\Delta_2\right)\left[\mathbf{ETT}^{-1}\left[\left(\frac{l}{r}\right)^{\alpha}\mathbf{ET}\left|\mathfrak{v}-\mathfrak{v}^*\right|\right]\right]\\
&\leq \left(\Delta_1+\Delta_2\right)\left[\mathbf{ET}^{-1}\left[\left(\frac{v}{r}\right)^{\alpha}\mathbf{ET}\left\|\mathfrak{u}-\mathfrak{u}^*\right\|\right]\right]\\
&= \left(\Delta_1+\Delta_2\right)\frac{u^{\alpha}}{\Gamma(1+\alpha)}\left\|\mathfrak{v}-\mathfrak{v}^*\right\|.
\end{aligned}$$

H is contraction as $0<\left(\Delta_1+\Delta_2\right)\dfrac{u^{\alpha}}{\Gamma(1+\alpha)}<1$. Different from the Banach fixed point theorem, the answer to (10) is unique [24]. □

Theorem 4.2. [25] *The ETDM-CF solution of* (10) *is unique when* $0<\left(\Delta_1+\Delta_2\right)(1-\alpha+\alpha v)<1$.

Proof. Since this proof is identical to Theorem 4.1's, it has been left out. □

Theorem 4.3. [5] *The ETDM-ABC solution of* (10) *is unique when,*

$$0<\left(\Delta_1+\Delta_2\right)\left(1-\alpha+\alpha\frac{\mathrm{t}^{\alpha}}{\Gamma(\alpha+1)}\right)<1.$$

Proof. This proof was left out since it is similar to Theorem 4.1. □

Theorem 4.4. *ETDM-C solution of* (10) *is convergent.*

Proof. Let $\mathfrak{v}_m=\sum_{r=0}^{m}\mathfrak{v}_r(\zeta,\mathrm{u})$. To prove that $\mathfrak{u}_m$ is a Cauchy sequence in F. Consider,

$$\begin{aligned}
\left\|\mathfrak{v}_m-\mathfrak{v}_k\right\| &= \max_{t\in J}\left|\mathfrak{v}_m-\mathfrak{v}_k\right|\\
&= \max_{u\in I}\left|\sum_{r=k+1}^{m}\mathfrak{v}_p\right|, n=1,2,3,\ldots\\
&\leq \max_{u\in I}\left|\mathbf{ET}^{-1}\left[\left(\frac{l}{s}\right)^{\alpha}\mathbf{ET}\left[\sum_{r=k+1}^{m}\left(\backslash\left(\mathfrak{u}_{r-1}\right)+\mathcal{E}\left(\mathfrak{v}_{rp-1}\right)\right)\right]\right]\right|\\
&= \max_{t\in J}\left|\mathbf{ET}^{-1}\left[\left(\frac{v}{s}\right)^{\alpha}\mathbf{ET}\left[\sum_{p=k}^{m-1}\left(\mathcal{L}\left(\mathfrak{v}_p\right)+\mathcal{N}\left(\mathfrak{u}_r\right)\right)\right]\right]\right|\\
&\leq \max_{u\in I}\left|\mathbf{ET}^{-1}\left[\left(\frac{v}{s}\right)^{\alpha}\mathbf{ET}\left[\left(\backslash\left(\mathfrak{v}_{m-1}\right)-\backslash\left(\mathfrak{v}_{k-1}\right)\right)\right]\right]\right|\\
&\quad+\max_{t\in J}\left|\mathbf{ET}^{-1}\left[\left(\frac{v}{s}\right)^{\alpha}\mathbf{NT}\left[\mathcal{N}\left(\mathfrak{u}_{m-1}\right)-\mathcal{E}\left(\mathfrak{u}_{n-1}\right)\right]\right]\right.
\end{aligned}$$

$$
\begin{aligned}
&\leq \delta_1 \max_{u\in I}\left|\mathbf{ET}^{-1}\left[\left(\frac{l}{s}\right)^{\alpha}\mathbf{ET}\left[\left(\mathcal{L}\left(\mathfrak{v}_{m-1}\right)-\backslash\left(\mathfrak{v}_{k-1}\right)\right)\right]\right]\right. \\
&\quad + \delta_2 \max_{\mathrm{t}\in I}\left|\mathbf{ET}^{-1}\left[\left(\frac{l}{s}\right)^{\alpha}\mathbf{ET}\left[\left(\mathcal{E}\left(\mathfrak{v}_{m-1}\right)-\mathcal{E}\left(\mathfrak{v}_{k-1}\right)\right)\right]\right]\right| \\
&= \left(\Delta_1+\Delta_2\right)\frac{\mathrm{u}^{\alpha}}{\Gamma(\alpha+1)}\left\|\mathfrak{u}_{m-1}-\mathfrak{v}_{k-1}\right\|.
\end{aligned}
$$

Let, $m=k+1$ then,

$$
\begin{aligned}
\left\|\mathfrak{v}_{k+1}-\mathfrak{v}_k\right\| &\leq \Delta\left\|\mathfrak{v}_k-\mathfrak{v}_{k-1}\right\| \\
&\leq \Delta^2\left\|\mathfrak{v}_{k-1}-\mathfrak{v}_{k-2}\right\| \\
&\leq \quad \vdots \\
&\leq \Delta^n\left\|\mathfrak{v}_1-\mathfrak{v}_0\right\|,
\end{aligned}
$$

where $\Delta=\left(\Delta_1+\Delta_2\right)\dfrac{\mathrm{t}^{\alpha}}{\Gamma(\alpha+1)}$. Similarly, we have,

$$
\begin{aligned}
\left\|\mathfrak{v}_m-\mathfrak{v}_k\right\| &\leq \left\|\mathfrak{v}_{k+1}-\mathfrak{v}_k\right\|+\left\|\mathfrak{v}_{k+2}-\mathfrak{v}_{k+1}\right\|+\ldots+\left\|\mathfrak{v}_m-\mathfrak{v}_{m-1}\right\| \\
&\leq \left(\Delta^k+\Delta^{k+1}+\ldots+\Delta^{m-1}\right)\left\|\mathfrak{k}_1-\mathfrak{k}_0\right\| \\
&\leq \Delta^k\left(\frac{1-\Delta^{m-n}}{1-\delta}\right)\left\|\mathfrak{k}_1\right\|.
\end{aligned}
$$

As $0<\Delta<1$, we get $1-\Delta^{m-k}<1$. Therefore,

$$
\left\|\mathfrak{v}_k-\mathfrak{v}_k\right\| \leq \frac{\Delta^k}{1-\Delta}\max_{\mathrm{t}\in J}\left\|\mathfrak{v}_1\right\|.
$$

Since $\left\|\mathfrak{v}_1\right\|<\infty$, as a result $k\to\infty$, then $\left\|\mathfrak{v}_m-\mathfrak{v}_k\right\|\to 0$. The series $\mathfrak{v}_m$ is convergent since $\mathfrak{v}_m$ is a Cauchy sequence in F. □

Theorem 4.5. [*11*] *ETDM-CF* (10) *has a convergent solution.*

Proof. Since this proof is similar to Theorem 4.4's, it has been left out. □

Theorem 4.6. [*8*] *ETDM-ABC* (10) *The solution is convergent.*

Proof. This proof has been left out as it is similar to Theorem 4.4's. □

5 Applications

Equation (1) has an approximate analytical solution, which is provided in this section. The effectiveness of the approach is demonstrated by the numerical results, which are also utilized to calculate its accuracy compared to precise and/or numerical answers found in previous research. The good performance and simple implementation of the findings are demonstrated by applying our approach. The results are shown in Tables 1, 2 and 3.

5.1 Tables and figures

Table 1: An analysis of the EFSKIE absolute errors with $c = 0.1$ and $\alpha = 1$.

t	η	ETDM-C	ETDM-CF	ETDM-ABC	q-HAM [42]
0.1	0.0	0	0	0	0
	3.0	0	0	0	$3.46944e-18$
	7.0	$1.73472e-18$	$1.73472e-18$	$1.73472e-18$	$5.20417e-18$
	10.0	$4.33681e-18$	$4.33681e-18$	$4.33681e-18$	$4.33681e-18$
0.5	0.0	0	0	0	0
	3.0	$3.46944e-18$	$3.46944e-18$	$3.46944e-18$	$3.46944e-18$
	7.0	$5.20417e-18$	$5.20417e-18$	$5.20417e-18$	0
	10.0	$2.16840e-18$	$2.16840e-18$	$2.16840e-18$	$2.16840e-18$
1.0	0.0	0	0	0	0
	3.0	$3.46944e-18$	$3.46944e-18$	$3.46944e-18$	0
	7.0	0	0	0	$5.20417e-18$
	10.0	$4.33681e-18$	$4.33681e-18$	$4.33681e-18$	$4.33681e-18$

Table 2: Comparison of the absolute errors of TFSKIE with $\alpha = 1$, and $c = 0.25$.

t	η	ETDM-C	ETDM-CF	ETDM-ABC	q-HAM [42]
0.1	0	$1.22634e-14$	$1.22634e-14$	$1.22634e-14$	$1.22679e-14$
	3.0	$5.88899e-15$	$5.88900e-15$	$5.88899e-15$	$5.89806e-15$
	7.0	$3.39521e-16$	$3.39514e-16$	$3.39521e-16$	$3.40006e-16$
	10.0	$2.93398e-16$	$2.93396e-16$	$2.93398e-16$	$2.91433e-16$
0.5	0	$7.66517e-12$	$7.66517e-12$	$7.66517e-12$	$7.66517e-12$
	3.0	$3.68154e-12$	$3.68154e-12$	$3.68154e-12$	$3.68154e-12$
	7.0	$2.12856e-13$	$2.12856e-13$	$2.12856e-13$	$2.12857e-13$
	10.0	$1.64157e-13$	$1.64157e-13$	$1.64157e-13$	$1.64160e-13$
1.	0	$1.22641e-10$	$1.22641e-10$	$1.22641e-10$	$1.22641e-10$
	3.0	$5.89071e-11$	$5.89071e-11$	$5.89071e-11$	$5.89071e-11$
	7.0	$3.41099e-12$	$3.41099e-12$	$3.41099e-12$	$3.41099e-12$
	10.0	$2.62408e-12$	$2.62408e-12$	$2.62408e-12$	$2.62408e-12$

Table 3: Comparison of the absolute errors of TFSKIE with $\alpha = 1$, and $c = 0.5$

t	η	**ETDM-C**	**ETDM-CF**	**ETDM-ABC**	**q-HAM** [42]
0.1	0	$1.31356e-05$	$1.31356e-05$	$1.31356e-05$	$1.31356e-05$
	3.0	$2.09024e-07$	$2.09024e-07$	$2.09024e-07$	$2.09024e-07$
	7.0	$4.54592e-08$	$4.54592e-08$	$4.54592e-08$	$4.54592e-08$
	10.0	$2.32576e-09$	$2.32576e-09$	$2.32576e-09$	$2.32576e-09$
0.5	0	$7.74067e-03$	$7.74067e-03$	$7.74067e-03$	$7.74067e-03$
	3.0	$2.32524e-05$	$2.32524e-05$	$2.32524e-05$	$2.32524e-05$
	7.0	$2.57421e-05$	$2.57421e-05$	$2.57421e-05$	$2.57421e-05$
	10.0	$1.31386e-06$	$1.31386e-06$	$1.31386e-06$	$1.31386e-06$
1.	0	$1.04808e-01$	$1.04808e-01$	$1.04808e-01$	$1.04808e-01$
	3.0	$1.02486e-03$	$1.02486e-03$	$1.02486e-03$	$1.02486e-03$
	7.0	$3.67202e-04$	$3.67202e-04$	$3.67202e-04$	$3.67202e-04$
	10.0	$1.87014e-05$	$1.87014e-05$	$1.87014e-05$	$1.87014e-05$

$$\begin{aligned}
\mathfrak{v}_0^c(\eta,\mathrm{t}) &= \frac{4}{3}b^2\left(2-3\tanh^2(b\eta)\right),\\
\mathfrak{v}_1^c(x,\mathrm{t}) &= -\frac{2048b^9\mathrm{t}^\beta\tanh(b\eta)\operatorname{sech}^2(b\eta)}{3\beta(\lambda+1)},\\
\mathfrak{v}_2^c(x,\mathrm{t}) &= \frac{524288b^{16}\mathrm{t}^{2\lambda}(\cosh(2b\eta)-2)\operatorname{sech}^4(b\eta)}{9\lambda(2\beta+1)},
\end{aligned}$$

substituting $\mathfrak{v}_0^c(\eta,t), \mathfrak{v}_1^c(\eta,\mathrm{t}),\ldots$ in (22). The series solution for ETDM-C is obtained as follows,

$$\begin{aligned}
\mathfrak{v}^a(\eta,\mathrm{t}) \approx{}& \frac{4}{3}b^2\left(2-3\tanh^2(b\eta)\right) - \frac{2048b^9\mathrm{t}^\beta\tanh(b\eta)\operatorname{sech}^2(b\eta)}{3\lambda(\beta+1)}\\
&+\frac{524288b^{16}\mathrm{t}^{2\beta}(\cosh(2b\eta)-2)\operatorname{sech}^4(b\eta)}{9\lambda(2\mu+1)}+\ldots.
\end{aligned}\tag{38}$$

The result is shown in Figure 1.

ETDM-CF: Utilizing the ETDM-CF, we produce the ensuing solutions,

$$\begin{aligned}
\mathfrak{v}_0^{CF}(\eta,\mathrm{t}) &= \frac{4}{3}b^2\left(2-3\tanh^2(b\eta)\right),\\
\mathfrak{v}_1^{CF}(\eta,\mathrm{t}) &= -\frac{1}{3}(2048)b^9(\beta(\mathrm{t}-1)+1)\tanh(b\eta)\operatorname{sech}^2(b\eta),\\
\mathfrak{v}_2^{CF}(\eta,\mathrm{t}) &= \frac{262144}{9}b^{16}\left(\beta^2\left(\mathrm{t}^2-4\mathrm{t}+2\right)+4\beta(\mathrm{t}-1)+2\right)\times(\cosh(2b\eta)-2)\operatorname{sech}^4(b\eta),
\end{aligned}$$

substituting $\mathfrak{u}_0^{CF}(\eta,\mathrm{t}), \mathfrak{v}_1^{CF}(\eta,\mathrm{t}),\ldots$ in (30), we obtain the ETDM-CF series solution as,

$$\begin{aligned}
\mathfrak{v}^{CF}(\eta,\mathrm{t}) \approx{}& \frac{4}{3}b^2\left(2-3\tanh^2(b\eta)\right)-\frac{1}{3}(2048)b^9(\beta(\mathrm{t}-1)+1)\tanh(b\zeta)\operatorname{sech}^2(b\eta)\\
&+\frac{262144}{9}b^{16}\left(\beta^2\left(\mathrm{t}^2-4\mathrm{t}+2\right)+4\beta(\mathrm{t}-1)+2\right)\times(\cosh(2b\eta)-2)\operatorname{sech}^4(b\eta)+\ldots.
\end{aligned}\tag{39}$$

The result is shown in Figure 2.

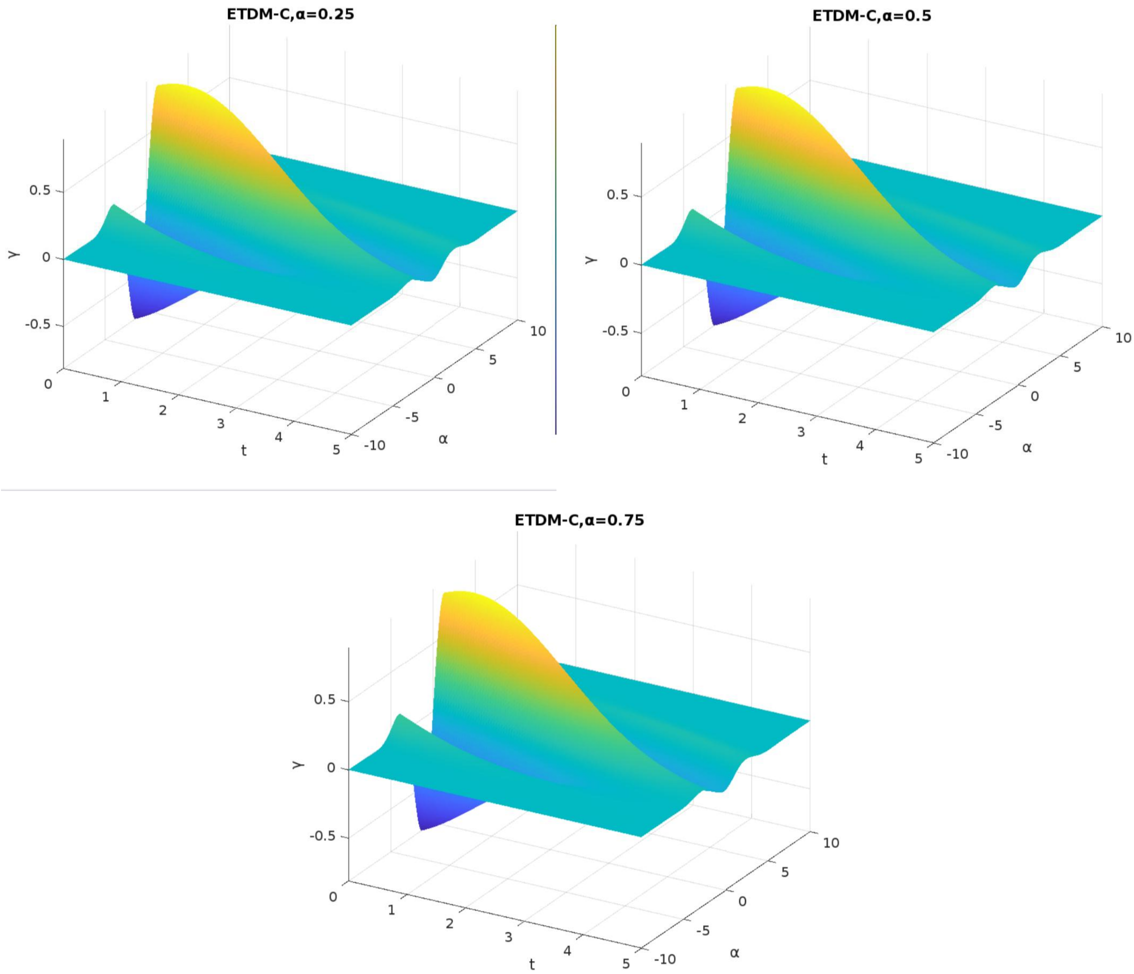

Figure 1: Numerical solution of ETDM-C with different α values and $c = 0.1$.

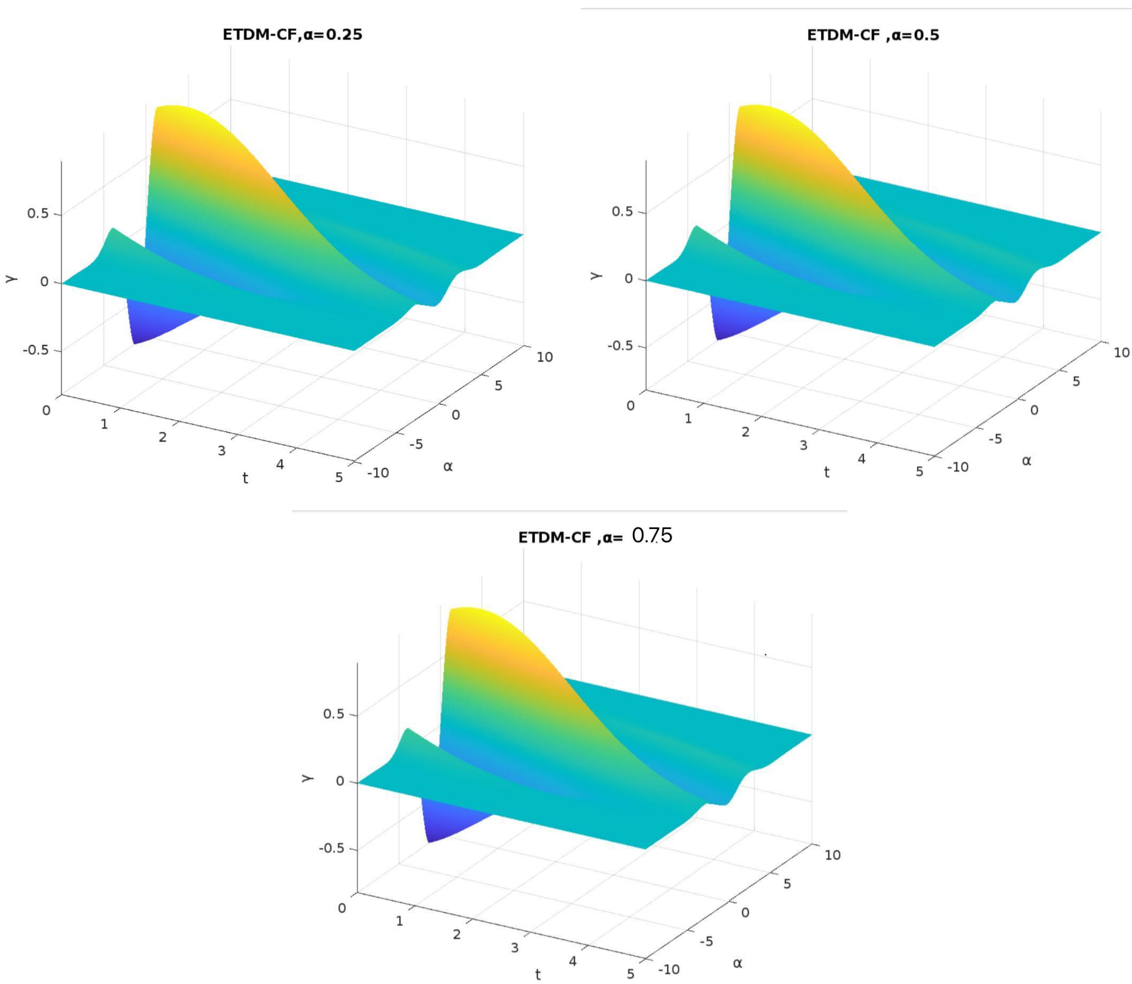

Figure 2: Numerical solution of ETDM-CF with different α values and $c = 0.1$.

ETDM-ABC: By employing the ETDM-ABC, we obtain the following successive solutions,

$$
\begin{aligned}
\mathfrak{v}_0^{ABC}(\eta, \mathrm{t}) &= \frac{4}{3}b^2\left(2 - 3\tanh^2(b\eta)\right),\\
\mathfrak{v}_1^{ABC}(\eta, \mathrm{t}) &= -\frac{2048}{3}b^9(\beta(\mathrm{t}-1)+1)\tanh(b\eta)\operatorname{sech}^2(b\eta),\\
\mathfrak{v}_2^{CF}(\eta, \mathrm{t}) &= \frac{262144}{9}b^{16}\left(\beta^2\left(\mathrm{t}^2 - 4\mathrm{t} + 2\right) + 4\beta\beta^{-}(\beta-1)t - 1) + 2\right) \times (\cosh(2b\eta) - 2)\operatorname{sech}^4(b\eta),
\end{aligned}
$$

substituting $\mathfrak{u}_0^{CF}(\eta, \mathrm{t}), \mathfrak{v}_1^{CF}(\eta, \mathrm{t}), \ldots$ in (30), we obtain the ETDM-CF series solution as,

$$
\begin{aligned}
\mathfrak{v}^{CF}(\eta, \mathrm{t}) \approx{}& \frac{4}{3}b^2\left(2 - 3\tanh^2(b\eta)\right) - \frac{1}{3}(2048)b^9(\beta(\mathrm{t}-1)+1)\tanh(b\zeta)\operatorname{sech}^2(b\eta)\\
&+ \frac{262144}{9}b^{16}\left(\beta^2\left(\mathrm{t}^2 - 4\mathrm{t} + 2\right) + 4\beta(\mathrm{t}-1) + 2\right) \times (\cosh(2b\eta) - 2)\operatorname{sech}^4(b\eta) + \ldots.
\end{aligned}
\tag{40}
$$

The result is shown in Figure 3.

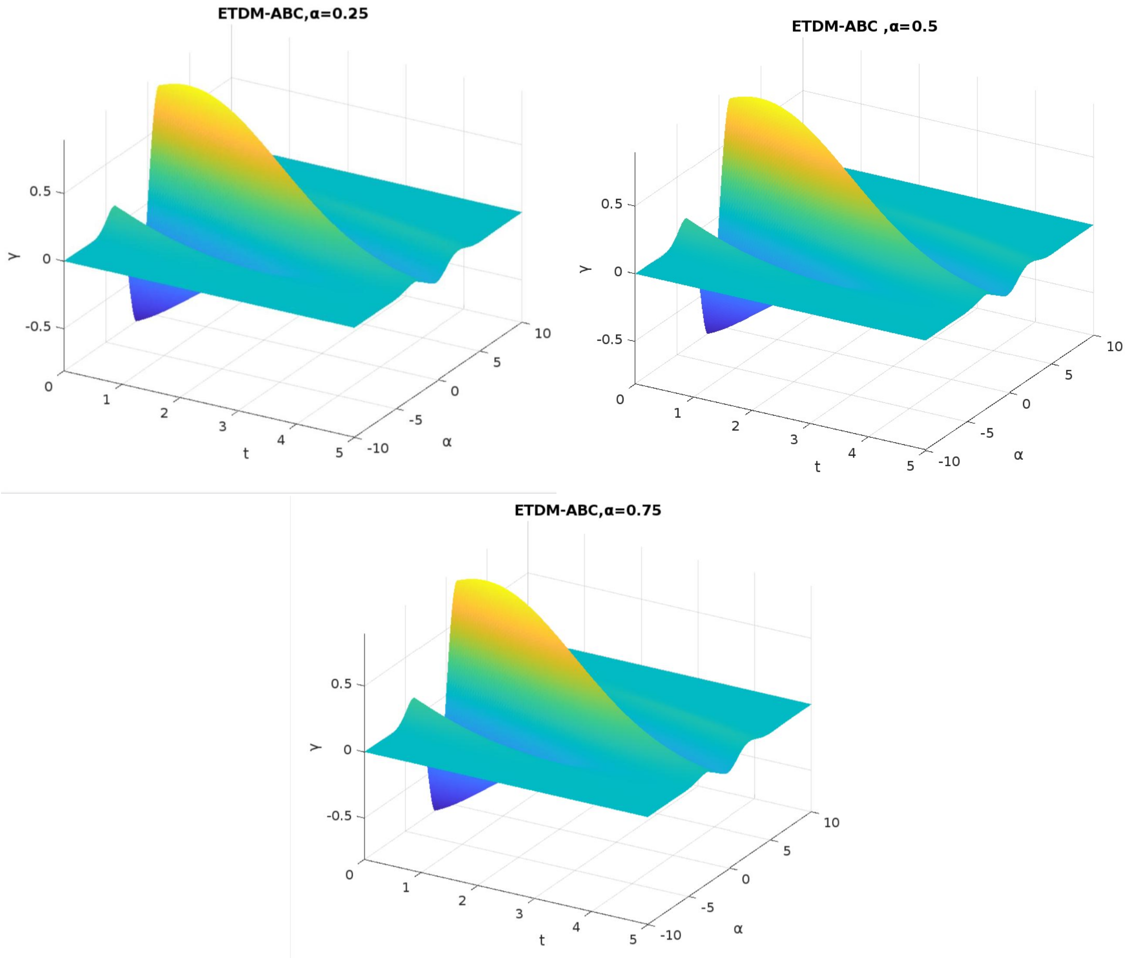

Figure 3: Numerical Solution of ETDM-ABC with different α values and $c = 0.1$.

6 Conclusions

In this work, we study TFSKIE using the Zaki transform with the benefit of Caputo, CF, and ABC derivatives. Where ETDM is a method that combines the Zaki transform and decomposition. Unlike the Adomian method, the proposed solution eliminates the need to calculate fractional derivatives and fractional integrals within the iteration process, which facilitates the estimation of the sequence. Numerical applications have demonstrated the effectiveness and accuracy of the proposed technique. To illustrate the theoretical perspective and visualize the dynamic behavior, the results of the present method are in good agreement with the proven results. The implemented methodology and the used fractional operator can be efficiently used in nonlinear fractional differential equations of variable order.

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Conflicts of Interest The authors declare no conflict of interest.

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